Table 2 Component stresses for final design of truss of Fig. 2 (iterative solution)

Component	σ_{Allow}, ksi	σ_{ij}, ksi Load condition 1	2	3	4
1	−9.710	0	0	−9.719	−9.719
2	−14.707	13.409	−13.409	−1.624	−14.709
3	−14.707	−13.409	13.409	−14.709	−1.624
4	−14.707	13.409	−13.409	−14.709	−1.624
5	−14.707	−13.409	13.409	−1.624	−14.709
6	−12.634	−3.598	3.598	−12.633	1.140
7	−12.634	3.598	−3.598	−12.633	1.140
8	−12.634	−3.598	3.598	1.140	−12.633
9	−12.634	3.598	−3.598	1.140	−12.633

Table 2. Computation time required to obtain this iterative solution on an IBM 360/91 was on the order of 0.8 sec.

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Transverse Vibrations Related to Stability of Curved Anisotropic Plates

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Introduction

THIS Note investigates the effect of inplane compressive loading on the natural frequencies of vibration of 45° angle-ply and cross-ply laminated composite plates. As shown by Lurie[1] for flat isotropic plates, the load, sufficiently large so as to cause the lowest natural frequency to be reduced to zero, corresponds to the critical buckling load. Hence, by this method, buckling information has been obtained for both lamina configurations as well. In addition, the effects of slight initial curvature and of various boundary conditions on frequency and buckling load are studied.

Analytical buckling investigations of laminated plates appear somewhat limited; however, there does exist work in this area. Typically, Ashton and Waddoups[2] have investigated the buckling of symmetrically layered angle-ply plates with loaded edges clamped and unloaded edges simply-supported. Whitney and Leissa[3,4] have investigated buckling of unsymmetrically layered angle-ply plates with two types of hinge boundaries. To date it appears no analytical buckling information is available for the cross-ply case.

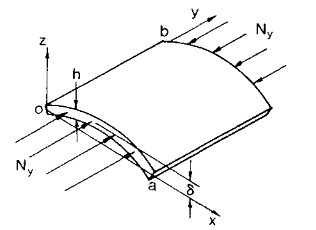

Fig. 1 Single constant curvature representation and inplane loading.

This study is restricted to those laminates symmetrically layered about the midplane or unsymmetrically layered but composed of a sufficient number of layers (usually 6 or more) so that the bending-extensional coupling present from unsymmetrical layering is eliminated. Without this restriction, buckling would not be characterized by a bifurcation but rather as a nonlinear (large displacement) response,[5] thus casting some doubt as to the validity of the results obtained from a linear analysis. In addition, the inplane loading is uniaxial and parallel to the axis of symmetry as shown in Fig. 1.

Analysis

Since the problem does not possess a closed form solution, the Ritz energy method was employed. The procedure, well established in the literature, requires the total energy expression expressed as a function of displacements. The strain-displacement relations used in this regard[6] are

$$\begin{aligned} \varepsilon_x &= u_{,x}{}^o - ww_{o,xx} - zw_{,xx} \\ \varepsilon_y &= v_{,y}{}^o - ww_{o,yy} - zw_{,yy} \\ \gamma_{xy} &= u_{,y}{}^o + v_{,x}{}^o - 2ww_{o,xy} - 2zw_{,xy} \end{aligned} \tag{1}$$

where u^o, v^o are midplane tangential displacements in the x, y directions, respectively, and w is the transverse displacement.

The initial shape, w_o, is assumed to be of the form $w_o = 4\delta(x/a - x^2/a^2)$ where δ is the maximum initial rise. With this form, a constant initial curvature, $w_{o,xx} = -8\delta/a^2$, is obtained. Note also that in (1) $w_{o,yy} = w_{o,xy} = 0$.

The strain energy expression for the plane stress situation assumed here is given by

$$\begin{aligned} U = \frac{1}{2}\int_o^a\int_o^b [&A_{11}(u_{,x}{}^o - ww_{o,xx})^2 + 2A_{12}(u_{,x}{}^o - ww_{o,xx})v_{,y}{}^o + \\ &2A_{16}(u_{,x}{}^o - ww_{o,xx})(u_{,y}{}^o + v_{,x}{}^o) + A_{22}(u_{,y}{}^o)^2 + \\ &2A_{26}(u_{,y}{}^o + v_{,x}{}^o)v_{,y}{}^o + A_{66}(u_{,y}{}^o + v_{,x}{}^o)^2 + D_{11}w_{,xx}{}^2 + \\ &2D_{12}w_{,xx}w_{,yy} + 4D_{16}w_{,xx}w_{,xy} + D_{22}w_{,yy}{}^2 + \\ &4D_{26}w_{,xy}w_{,yy} + 4D_{66}w_{,xy}{}^2 + N_y w_{,y}{}^2]\,dx\,dy \end{aligned} \tag{2}$$

where A_{ij}, D_{ij} are constitutive coefficients, as defined in Ref. 3, and N_y is the inplane force per unit length as shown in Fig. 1. The kinetic energy, T, is given by

$$T = \frac{1}{2}\int_o^a\int_o^b \rho h[(u_{,t}{}^o)^2 + (v_{,t}{}^o)^2 + (w_{,t})^2]\,dx\,dy \tag{3}$$

Table 1 Boundary conditions

	At $x = 0, a$	At $y = 0, b$
HFT	$w = u^o = M_x = N_{xy} = 0$	$w = u^o = M_y = N_{xy} = 0$
HFN	$w = v^o = M_x = N_x = 0$	$w = u^o = M_y = N_y = 0$
HR	$w = u^o = v^o = M_x = 0$	$w = u^o = v^o = M_y = 0$
CL	$w = w_{,x} = u^o = v^o = 0$	$w = w_{,y} = u^o = v^o = 0$

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where ρ is the mass per unit volume and h is the total thickness.

The Ritz procedure for dynamic analysis requires the use of Hamilton's principle. Accordingly,

$$\delta \Pi = 0 \tag{4}$$

where $\Pi = U - T$. In addition, displacement functions are assumed in the form

$$\begin{aligned} w &= \sum_{i=1}^{m} \sum_{j=1}^{n} E_{ij} \phi_{wi}(x) \theta_{wj}(y) \\ u^o &= \sum_{i=1}^{m} \sum_{j=1}^{n} F_{ij} \phi_{ui}(x) \theta_{uj}(y) \\ v^o &= \sum_{i=1}^{m} \sum_{j=1}^{n} G_{ij} \phi_{vi}(x) \theta_{vj}(y) \end{aligned} \tag{5}$$

where $\phi(x)$, $\theta(y)$ are functions satisfying the geometric boundary conditions along $x = 0, a$ and $y = 0, b$ respectively, and E_{ij}, F_{ij}, G_{ij} are undetermined coefficients. Now since $\Pi = \Pi(u^o, v^o, w)$ and u^o, v^o and w are functions of E_{ij}, F_{ij} and G_{ij}, the variational problem of (4) may be substituted by

$$\partial \Pi / \partial (E_{kl}, F_{kl}, G_{kl}) = 0 \tag{6}$$

where $k = 1, \ldots, m$ and $l = 1, \ldots, n$. Now substituting (2), (3) and (5) into (6) and carrying out the differentiation, there results a system of $3 \cdot m \cdot n$ algebraic equations in the undetermined coefficients. Solving the resulting natural frequency eigenvalue problem yields the desired relationship between natural frequencies and the magnitude of inplane loading.

Due to the presence of initial curvature, it is necessary to specify boundary conditions for the inplane as well as the transverse displacements. In this regard, three types of hinge boundaries, as well as a fully restrained boundary are defined as shown in Table 1. The designations HFT, HFN, HR, and CL denote hinge-free-tangential, hinge-free-normal, hinge-restrained and clamped respectively. Beam mode shape functions, tabulated in the literature,[7] were chosen for the functions $\phi(x)$ and $\theta(y)$. Excellent convergence of solutions was demonstrated by increasing m and n in (5) and observing the asymptotic nature of the solution. The final data was obtained with $m = n = 5$.

Discussion

The case considered was a square ($a = b$) plate with three degrees of initial curvature ($\delta/a = 0, 0.01, 0.02$) for each boundary condition and lamina configuration. The material properties chosen

$$E_L/E_T = 40, \quad G_{LT}/E_T = 1, \quad \nu_{LT} = 0.25$$

were typical of high modulus graphite-epoxy laminates.

Figure 2 is typical of the results obtained. Note the apparent linear relationship existing between the square of the frequency and the inplane load, a relation which may in fact be proven linear for isotropic flat plates.[1] The abrupt change in slope (present in all cases except HFN cross-ply) when initial curvature is present occurs because the rate of change of frequency with inplane loading is greater for higher modes of vibration than for the fundamental, and also the curvature does not have as great an effect on increasing the higher mode frequencies as it does on the fundamental. Hence, a plot of the second lowest mode frequency intersects a plot of the lowest mode frequency, and if only the lowest frequency is plotted, as is the case here, the graph will show a discontinuity at that intersection point.

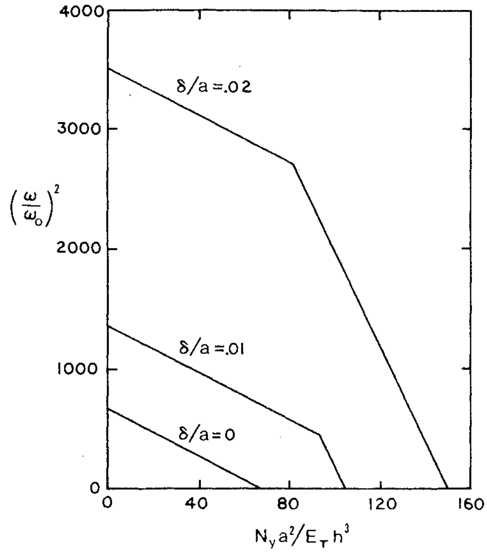

Fig. 2 Lowest natural frequency as a function of inplane loading and initial curvature for a square 45° angle-ply plate with HFT boundary; $\omega_o = (E_T h^3/\rho a^4)^{1/2}$.

The stiffening effect due to initial curvature shows itself in the increase in buckling load as curvature increases as shown in Table 2. The slope discontinuity, however, indicates that this increase is not in direct proportion to the increase in the fundamental frequency with no inplane loading, an assumption which if made would lead to gross inaccuracies.

For the flat plate cases ($\delta/a = 0$) all three hinge boundary conditions become the usual simple-support condition since the inplane restraints become unnecessary. Thus, the buckling loads are identical for the three angle-ply hinges and identical, but different from the angle-ply value, for the three cross-ply hinges. Note that for the flat case the buckling load for the 45° angle-ply appears significantly higher than its cross-ply counterpart for the hinge cases but is almost identical for the clamped case.

Table 2 Nondimensional buckling loads and fundamental (no load) frequencies for various boundary conditions and initial curvatures

	$N_y a^2/E_T h^3$ at $\omega/\omega_o = 0$					
	45° Angle-ply			Cross-ply		
	$\delta/a=0$	$\delta/a=0.01$	$\delta/a=0.02$	$\delta/a=0$	$\delta/a=0.01$	$\delta/a=0.02$
HFT	67	105	150	37	112	163
HFN	67	96	121	37	44	61
HR	67	99	130	37	102	167
CL	131	152	174	133	179	234
	$(\omega/\omega_o)^2$ at $N_y = 0$					
HFT	667	1381	3523	370	1639	5430
HFN	667	995	1977	370	428	602
HR	667	1234	2901	370	1430	3183
CL	1662	2153	3607	1770	2678	5315

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